선 형 변 환

2장 Continuous System

1. Linearity

A system is **Linear** ⟷ (if and only if) it satisfies the principle of **homogeneity** and **additivity** ⇒ **Superposition**

Homogeneity

$x_1(t) \rightarrow$ Linear System $\rightarrow y_1(t)$ ⇒ $x(t) = C_1 x_1(t) \rightarrow$ Linear System $\rightarrow y(t) = C_1 y_1(t)$

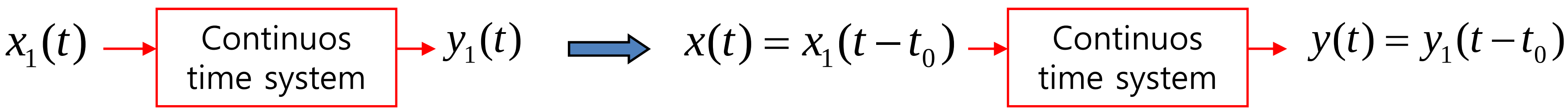
Additivity

$x_1(t), x_2(t) \rightarrow$ Linear System $\rightarrow y_1(t), y_2(t)$ ⇒ $x(t) = x_1(t) + x_2(t) \rightarrow$ Linear System $\rightarrow y(t) = y_1(t) + y_2(t)$

Superposition

$x(t) = C_1 x_1(t) + C_2 x_2(t) \rightarrow$ Linear System $\rightarrow y(t) = C_1 y_1(t) + C_2 y_2(t)$

- Time Invariance



- Linear Time Invariance (LTI)

When a system if both linear and time invariant,
it is called a linear time invariant (LTI) system

- Causality

its current output depends on past and current inputs
but not on future input.

- Stability

if the system input is bounded,
and if the system output is also bounded, it is called that
the system is stable (BIBO)

HomeWork #1)

$$f_1(t) = 4t$$
$$f_2(t) = e^{-t}$$
$$f_3(t) = \ln t$$
$$f_4(t) = 4t + 1$$
$$f_5(t) = 2t^2$$
$$f_6(t) = 2t^2 + 1$$
$$f_7(t) = \sin(2\pi t + 1)$$

위 함수들이 선형인지 비선형인지 판단하고 그 이유를 증명하시오.

2. 미분방정식의 종류

미분방정식

$$\frac{dy}{dx} = f(x)$$

상미분방정식
(Ordinary Differential Equation)

$$F(x, y, y', y'', \cdots, c) = 0$$

편미분방정식
(Partial Differential Equation)

$$F(x_1, x_2, \cdots, \frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2}, \cdots, \frac{\partial^2 y}{\partial x_1 \partial x_2}, \cdots) = 0$$

전미분방정식
(Total Differential Equation)

$$F(x, y, c) = 0$$

상미분방정식
(Ordinary Differential Equation)

$$F(x, y, y', y'', \cdots, c) = 0$$

종속변수와 그 도함수에 관하여 모두 1차 → **선형미방**

$$\cos x \frac{dy}{dx} + x^2 y = 0, \quad \frac{d^2 y}{dx^2} + x^2 y = 0$$

종속변수 $y,\ y',\ y''$에 대해 선형이므로

비선형미방 → $y\frac{dy}{dx} = x, \quad \frac{dy}{dx} + \cos y = 0, \quad \frac{d^2 y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 1$

일반적인 선형 미분방정식 $a_n(x)\frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x)$

$f(x) = 0$ **제차 미분방정식**
(Homogeneous DE)

3. 변수 분리형 미분방정식

$$g(y)\frac{dy}{dx} = f(x)$$

$$d(y)dy = f(x)dx$$

$$\int d(y)dy = \int f(x)dx + c$$

일반해 : 적분상수를 포함한 해

특수해 : 초기조건에 의해 적분상수가 정해지는 해

$y(x_0)$

Example) $\dfrac{dy}{dx} + y = 0$

$$\frac{1}{y}\frac{dy}{dx} = -1 \quad \rightarrow \quad \frac{1}{y}dy = -dx$$

$$\int \frac{1}{y}dy = \int -dx \quad \rightarrow \quad \ln|y| = -x + c$$

$$y = e^{-x+c} = e^c e^{-x}, \qquad \therefore y = Ke^{-x} \quad (K = e^c)$$

일반해

$if \quad y(0) = 5$

$y = 5e^{-x}$ **특수해**

4. Homogeneous Linear Equations

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

$$ay'' + by' + cy = 0$$

$\downarrow$ $\leftarrow$ $y = e^{mx} \leftrightarrow y' = me^{mx}, \quad y'' = m^2 e^{mx}$

$$e^{mx}(am^2 + bm + c) = 0$$

보조방정식
(Auxiliary Equation)

$$\boxed{am^2 + bm + c = 0} \rightarrow m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$m_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad m_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$ay'' + by' + cy = 0$$

$$b^2 - 4ac > 0 \longrightarrow y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

$$b^2 - 4ac = 0 \longrightarrow y = C_1 e^{m_1 x} + C_2 x e^{m_2 x}$$

$$b^2 - 4ac < 0 \longrightarrow y = C_1 e^{(\alpha + j\beta)x} + C_2 e^{(\alpha - j\beta)x}$$

$$\left[e^{j\theta} = \cos\theta + j\sin\theta \right]$$

$$y = C_1 e^{\alpha x} \cos\beta x + C_2 e^{\alpha x} \sin\beta x$$

Example)

$$2y'' - 5y' - 3y = 0$$
$$\rightarrow 2m^2 - 5m - 3 = 0 \rightarrow (2m+1)(m-3)$$
$$\rightarrow y = C_1 e^{-\frac{1}{2}x} + C_2 e^{3x}$$

$$y'' - 10y' + 25y = 0$$
$$\rightarrow m^2 - 10m + 25 = 0 \rightarrow (m-5)^2$$
$$\rightarrow y = C_1 e^{5x} + C_2 x e^{5x}$$

$$y'' + 4y' + 7y = 0$$
$$\rightarrow m^2 + 4m + 7 = 0 \rightarrow m_1 = -2 + \sqrt{3}j, \quad m_2 = -2 - \sqrt{3}j$$
$$\rightarrow y = e^{-2x}\left(C_1 \cos\sqrt{3}x + C_2 \sin\sqrt{3}x\right)$$

Example)

Solve $y''' + 3y'' - 4y = 0$.

SOLUTION It should be apparent from inspection of $m^3 + 3m^2 - 4 = 0$ that one root is $m_1 = 1$ and so $m - 1$ is a factor of $m^3 + 3m^2 - 4$. By division we find

$$m^3 + 3m^2 - 4 = (m - 1)(m^2 + 4m + 4) = (m - 1)(m + 2)^2,$$

and so the other roots are $m_2 = m_3 = -2$. Thus the general solution is

$$y = c_1 e^x + c_2 e^{-2x} + c_3 x e^{-2x}.$$

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Example)

Solve $\dfrac{d^4y}{dx^4} + 2\dfrac{d^2y}{dx^2} + y = 0.$

SOLUTION The auxiliary equation $m^4 + 2m^2 + 1 = (m^2 + 1)^2 = 0$ has roots $m_1 = m_3 = i$ and $m_2 = m_4 = -i$. Thus from Case II the solution is

$$y = C_1e^{ix} + C_2e^{-ix} + C_3xe^{ix} + C_4xe^{-ix}.$$

By Euler's formula the grouping $C_1e^{ix} + C_2e^{-ix}$ can be rewritten as $c_1 \cos x + c_2 \sin x$ after a relabeling of constants. Similarly, $x(C_3e^{ix} + C_4e^{-ix})$ can be expressed as $x(c_3 \cos x + c_4 \sin x)$. Hence the general solution is

$$y = c_1 \cos x + c_2 \sin x + c_3x \cos x + c_4x \sin x.$$

HomeWork #2)

$$4y'' + y' = 0$$

$$y'' - 36y = 0$$

$$y'' - y' - 6y = 0$$

$$y'' - 10y' + 25y = 0$$

$$12y'' + 8y' + 16y = 0$$

$$y'' + 4y' - y = 0$$

5. Non-Homogeneous Linear Equations

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x)$$

해법

1. 보조해를 찾는다. y_c

2. 특수해를 찾는다. y_p

$$\boxed{y = y_c + y_p}$$

Example 1 General Solution Using Undetermined Coefficients

Solve $y'' + 4y' - 2y = 2x^2 - 3x + 6.$ (2)

SOLUTION **Step 1.** We first solve the associated homogeneous equation $y'' + 4y' - 2y = 0$. From the quadratic formula we find that the roots of the auxiliary equation $m^2 + 4m - 2 = 0$ are $m_1 = -2 - \sqrt{6}$ and $m_2 = -2 + \sqrt{6}$. Hence the complementary function is

$$y_c = c_1 e^{-(2+\sqrt{6})x} + c_2 e^{(-2+\sqrt{6})x}.$$

$$y'' + 4y' - 2y = 2x^2 - 3x + 6.$$

Step 2. Now, since the function $g(x)$ is a quadratic polynomial, let us assume a particular solution that is also in the form of a quadratic polynomial:

$$y_p = Ax^2 + Bx + C.$$

We seek to determine *specific* coefficients A, B, and C for which y_p is a solution of (2). Substituting y_p and the derivatives $y_p' = 2Ax + B$ and $y_p'' = 2A$ into the given differential equation (2), we get

$$\begin{aligned} y_p'' + 4y_p' - 2y_p &= 2A + 8Ax + 4B - 2Ax^2 - 2Bx - 2C \\ &= 2x^2 - 3x + 6. \end{aligned}$$

Since the coefficients of like powers of

$$y_p'' + 4y_p' - 2y_p = 2A + 8Ax + 4B - 2Ax^2 - 2Bx - 2C$$
$$= 2x^2 - 3x + 6.$$

equal

$$\boxed{-2a}\; x^2 + \boxed{8A - 2B}\; x + \boxed{2A + 4B - 2C} = 2x^2 - 3x + 6.$$

That is,

$$-2A = 2, \qquad 8A - 2B = -3, \qquad 2A + 4B - 2C = 6.$$

Solving this system of equations leads to the values $A = -1$, $B = -5/2$, and $C = -9$. Thus a particular solution is

$$y_p = -x^2 - \frac{5}{2}x - 9.$$

Step 3. The general solution of the given equation is

$$y = y_c + y_p = c_1 e^{-(2+\sqrt{6})x} + c_2 e^{(-2+\sqrt{6})x} - x^2 - \frac{5}{2}x - 9.$$

Example 2 Particular Solution Using Undetermined Coefficients

Find a particular solution of $y'' - y' + y = 2\sin 3x$.

SOLUTION A natural first guess for a particular solution would be $A \sin 3x$. But since successive differentiations of $\sin 3x$ produce $\sin 3x$ *and* $\cos 3x$, we are prompted instead to assume a particular solution that includes both of these terms:

$$y_p = A\cos 3x + B\sin 3x.$$

Differentiating y_p and substituting the results into the differential equation gives, after regrouping,

$$y_p'' - y_p' + y_p = (-8A - 3B)\cos 3x + (3A - 8B)\sin 3x = 2\sin 3x$$

or

equal

$$\boxed{-8A - 3B}\cos 3x + \boxed{3A - 8B}\sin 3x = 0\cos 3x + 2\sin 3x.$$

From the resulting system of equations,

$$-8A - 3B = 0, \qquad 3A - 8B = 2,$$

we get $A = 6/73$ and $B = -16/73$. A particular solution of the equation is

$$y_p = \frac{6}{73}\cos 3x - \frac{16}{73}\sin 3x.$$

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Example 3 Forming y_p by Superposition

Solve $y'' - 2y' - 3y = 4x - 5 + 6xe^{2x}$. (3)

SOLUTION **Step 1.** First, the solution of the associated homogeneous equation $y'' - 2y' - 3y = 0$ is found to be $y_c = c_1e^{-x} + c_2e^{3x}$.

Step 2. Next, the presence of $4x - 5$ in $g(x)$ suggests that the particular solution includes a linear polynomial. Furthermore, since the derivative of the product xe^{2x} produces $2xe^{2x}$ and e^{2x}, we also assume that the particular solution includes both xe^{2x} and e^{2x}. In other words, g is the sum of two basic kinds of functions:

$$g(x) = g_1(x) + g_2(x) = \textit{polynomial} + \textit{exponentials}.$$

Correspondingly, the superposition principle for nonhomogeneous equations (Theorem 3.7) suggests that we seek a particular solution

$$y_p = y_{p_1} + y_{p_2},$$

where $y_{p_1} = Ax + B$ and $y_{p_2} = Cxe^{2x} + Ee^{2x}$. Substituting

$$\boxed{y_p = Ax + B + Cxe^{2x} + Ee^{2x}}$$

$$y_p = Ax + B + Cxe^{2x} + Ee^{2x}$$

into the given equation (3) and grouping like terms gives

$$y_p'' - 2y_p' - 3y_p = -3Ax - 2A - 3B - 3Cxe^{2x} + (2C - 3E)e^{2x} = 4x - 5 + 6xe^{2x}. \quad (4)$$

From this identity we obtain the four equations

$$-3A = 4, \quad -2A - 3B = -5, \quad -3C = 6, \quad 2C - 3E = 0.$$

The last equation in this system results from the interpretation that the coefficient of e^{2x} in the right member of (4) is zero. Solving, we find $A = -\frac{4}{3}$, $B = \frac{23}{9}$, $C = -2$, and $E = -\frac{4}{3}$. Consequently,

$$y_p = -\frac{4}{3}x + \frac{23}{9} - 2xe^{2x} - \frac{4}{3}e^{2x}.$$