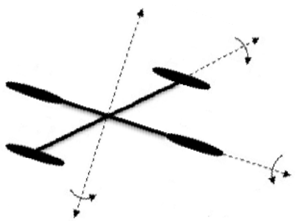
3-3. 상수계수를 가진 동차 선형 미분방정식



- 1계 동차 선형 미분방정식의 해

$$y' + ay = 0 \quad \text{solution} \rightarrow \quad y = c_1 e^{-ax}$$

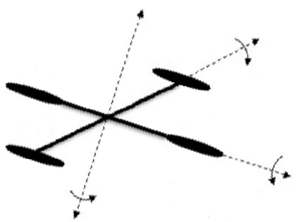
- 2계 동차 선형 미분방정식의 해

$$ay'' + by' + cy = 0.$$

$$am^2 e^{mx} + bme^{mx} + ce^{mx} = 0, \quad e^{mx}(am^2 + bm + c) = 0$$

$$am^2 + bm + c = 0.$$

$$m_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$



Case I: Distict Roots $(D = b^2 - 4ac > 0)$

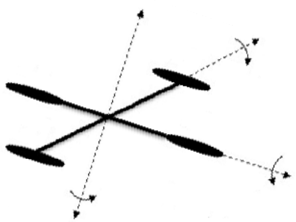
$$y_1 = e^{m_1 x},\ y_2 = e^{m_2 x} \rightarrow \ y = c_1 e^{m_1 x} + c_2 e^{m_2 x}.$$

Case II: Repeated Real Roots $(D = b^2 - 4ac = 0)$

$$y_2 = e^{m_1 x} \int \frac{e^{2m_1 x}}{e^{2m_1 x}} dx = e^{m_1 x} \int dx = x e^{m_1 x}. \rightarrow \ y = c_1 e^{m_1 x} + c_2 x e^{m_1 x}.$$

Case III: Conjugate Complex Roots $(D = b^2 - 4ac < 0)$

$$y = C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x}.$$



Use Euler's firmula

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

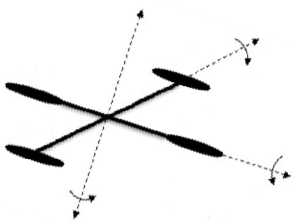
$$e^{i\beta x} = \cos\beta x + i\sin\beta x \quad \text{and} \quad e^{-i\beta x} = \cos\beta x - i\sin\beta x,$$

$$e^{i\beta x} + e^{-i\beta x} = 2\cos\beta x \quad \text{and} \quad e^{i\beta x} - e^{-i\beta x} = 2i\sin\beta x.$$

$$y_1 = e^{(\alpha+i\beta)x} + e^{(\alpha-i\beta)x} = e^{\alpha x}(e^{i\beta x} + e^{-i\beta x}) = 2e^{\alpha x}\cos\beta x \ \ (C_1 = 1,\ C_2 = 1)$$

$$y_2 = e^{(\alpha+i\beta)x} - e^{(\alpha-i\beta)x} = e^{\alpha x}(e^{i\beta x} - e^{-i\beta x}) = 2ie^{\alpha x}\sin\beta x \ \ (C_1 = 1,\ C_2 = -1)$$

$$\therefore y = c_1e^{\alpha x}\cos\beta x + c_2e^{\alpha x}\sin\beta x = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x).$$



Example 1 Second-Order DEs

(a) $2y'' - 5y' - 3y = 0$ (b) $y'' - 10y' + 25y = 0$ (c) $y'' + 4y' + 7y = 0$

Solution

(a) $2m^2 - 5ym - 3 = (2m+1)(m-3), \quad m_1 = -\tfrac{1}{2}, \; m_2 = 3.$

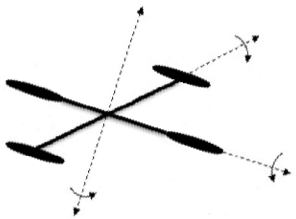
$$y = c_1 e^{-x/2} + c_2 e^{3x}.$$

(b) $m^2 - 10m + 25 = (m-5)^2, \quad m_1 = m_2 = 5.$

$$y = c_1 e^{5x} + c_2 x e^{5x}.$$

(c) $m^2 + 4m + 7 = 0, \quad m_1 = -2 + \sqrt{3}i, \quad m_2 = -2 - \sqrt{3}i.$

$$\alpha = -2, \; \beta = \sqrt{3} \;\rightarrow\; y = e^{-2x}(c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x).$$



Example 2 An Initial-Value Problem

$$4y'' + 4y' + 17y = 0, \quad y(0) = -1, \quad y'(0) = 2.$$

Solution

$$4m^2 + 4m + 17 = 0, \qquad m_1 = -(1/2) + 2i, \qquad m_2 = -(1/2) - 2i.$$

$$\alpha = -(1/2), \; \beta = 2$$

$$y = e^{-x/2}(c_1 \cos 2x + c_2 \sin 2x)$$

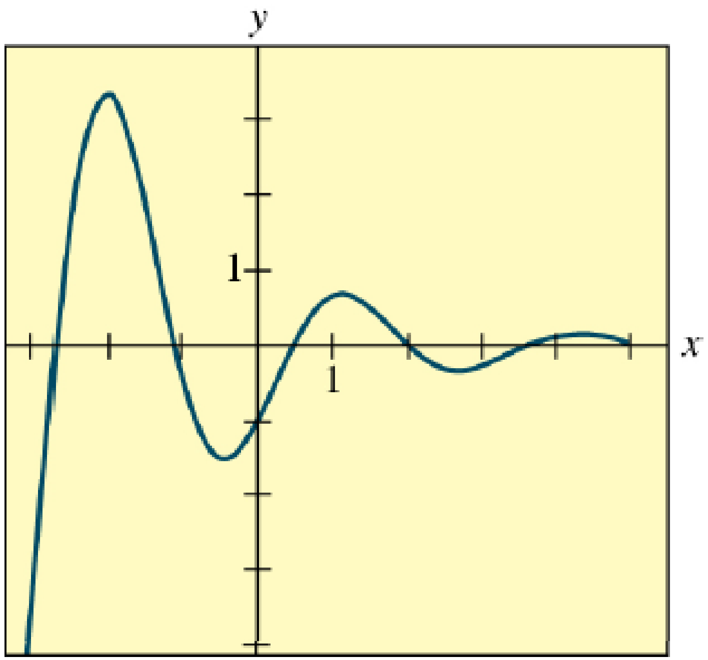
$$y(0) = -1$$

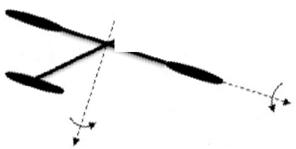
$$e^0(c_1 \cos 0 + c_2 \sin 0) = -1 \;\rightarrow\; c_1 = -1$$

$$y'(0) = 2$$

$$2c_2 + (1/2) = 2 \;\rightarrow\; c_2 = 3/4$$

$$\rightarrow \quad y = e^{-x/2}\left(-\cos 2x + \frac{3}{4}\sin 2x\right)$$





Two Equations Worth Knowing

$$y'' + k^2 y = 0$$

$$m^2 + k^2 = 0 \;\rightarrow\; m_{1,2} = 0 \pm ik \;\rightarrow\; \alpha = 0,\;\; \beta = k$$

$$y = c_1 \cos kx + c_2 \sin kx.$$

$$y'' - k^2 y = 0$$

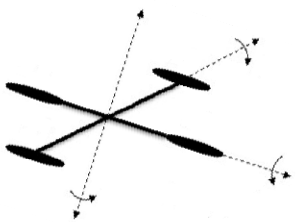
$$m^2 - k^2 = 0 \;\rightarrow\; m_{1,2} = \pm k$$

$$y = c_1 e^{kx} + c_2 e^{-kx}.$$

$$c_1 = 1/2,\;\; c_2 = 1/2 \;\rightarrow\; y_1 = \frac{e^{kx} + e^{-kx}}{2} = \cosh kx$$

$$c_1 = 1/2,\;\; c_2 = -1/2 \;\rightarrow\; y_1 = \frac{e^{kx} - e^{-kx}}{2} = \sinh kx$$

$$y = c_1 \cosh kx + c_2 \sinh kx.$$



Higher-Order Equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0,$$

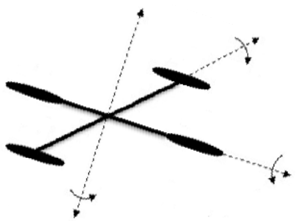
$$a_n m^n + a_{n-1} m^{n-1} + \cdots + a_2 m^2 + a_1 m + a_0 = 0.$$

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}.$$

k – roots

$$e^{m_1 x}, x e^{m_1 x}, x^2 e^{m_1 x}, \ldots, x^{k-1} e^{m_1 x}$$

$$y = c_1 e^{m_1 x} + c_2 x e^{m_1 x} + c_3 x^2 e^{m_1 x} + \cdots + c_k x^{k-1} e^{m_1 x}.$$



Example 3 Third-Order DE

$$y''' + 3y'' - 4y = 0.$$

Solution

$$m^3 + 3m^2 - 4 = (m-1)(m^2 + 4m + 4) = (m-1)(m+2)^2 = 0$$

$m = 1$, $m = -2$ (roots)

$$y = c_1 e^{x} + c_2 e^{-2x} + c_3 x e^{-2x}.$$

Example 4 Fourth-Order DE

$$\frac{d^4y}{dx^4} + 2\frac{d^2y}{dx^2} + y = 0.$$

Solution

$$m^4 + 2m^2 + 1 = (m^2+1)^2 = 0 \rightarrow m = i \text{ (roots)}, \quad m = -i \text{ (roots)}$$

$$y = C_1e^{ix} + C_2e^{-ix} + C_3xe^{ix} + C_4xe^{-ix}.$$

$$y = c_1\cos x + c_2\sin x + c_3x\cos x + c_4x\sin x.$$